THE HYPERSONIC AREA RULE

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THE HYPERSONIC AREA RULE

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The hypersonic area rule has been formulated in a work [1] and in a hypothesis [2] that all the mass of a gas is concentrated in an infinitely thin layer contiguous to the shock wave. According to that rule when thin non-axially-symmetrical blunt bodies with the same bluntness drag values and the same laws of transverse area change in the direction of the stream, are subjected to flow, the shock-wave surfaces, the laws of pressure change, and consequently also the forces of resistance acting on the body coincide, and in this process the surfaces of the shock waves have axial symmetry.

1. Determining the results' bounds of applicability [1]. As an example of applying the hypersonic area rule [1] let us construct a body equivalent to a thin cone of revolution, i.e, having the same bluntness drag values and the same law of cross-sectional area change in the stream direction as a cone. The cross-section of the body is assumed to be elliptical, the larger semiaxis of which equals the radius of the shock wave and the area of which equals the area of the cross-section of the cone of revolution (Fig. 1). In other words the eccentricity of the ellipse in each section has the largest possible value compatible with the demand (condition 3 in work [1]) that the

body must not exceed the volume delimited by the shock wave.

As was noted in the work [1] the area rule may be combined with the law of similarity in the flow about thin blunt bodies [2], as a result of which dimensionless values characterizing the flow are defined at a fixed value of the index of adiabatic curve $\varkappa$ by two dimensionless parameters: the known parameter of similarity of flow about thin blunt bodies $K = M_\infty \tau$ and the parameter $K_1 = \pi/2c_xS)^{1/2}L\tau^2$, characterizing the effect of bluntness and equal in order of size to the square root of the ratio of body drag to bluntness drag. Here $\tau \sim S_*^{1/2}/L$ is a small dimensionless value characterizing the thickness of the body; S_* is a certain representative area of the body's cross-section; L is the length of the body; c_x, S are coefficient of bluntness drag and the area of the mid-section of the bluntness [surface area per unit depth] respectively.

The shape of the bluntness is non-essential in the assumption [1] that the effect of bluntness may be replaced by the effect of a blast at the forward point of the body with an energy equal to bluntness drag. In view of this the area of bluntness S is introduced into the expression for K_1 instead of the diameter of the bluntness. Let us suppose that the Mach number M_∞ of the free stream equals infinity. Then when $\varkappa$ is fixed the dimensionless variables will depend upon the single parameter K_1.

Figure 2 shows in the form of curves (for the case of K = 1.4) the relationships of the values k (ratio of the major to the minor semiaxis) and $(X - X_0)/X_0$ (ratio of body drag without bluntness drag to bluntness drag) in the function $K_1 = (\pi/2c_xS)^{1/2}L \tan^2 \alpha$ where α is half the apex angle of the circular cone. The shape of the shock wave was defined from the solution of the problem of flow around a thin blunted cone in accordance with Chernyy [2]. It is sensible to use the area rule when $X/X_0 \geq 1.1$, which, as follows from Fig. 2, corresponds to $K_1 \geq 0.1$. At lesser values of K_1 the body drag is practically determined by the value of the bluntness drag. At large values of K_1 the area rule becomes invalid when k approaches unity (more accurately when $k-1 \sim (\varkappa-1)/(\varkappa+1)$ [1]), which occurs approximately at $K_1 = 1.2$.

Thus, the range of application of the area rule lies within the limits $0.1 \leq K_1 \leq 1.2$. Moreover the ellipse in the cross-section of the body may have an amply extended shape differing from a circle ($13 \geq k \geq 1.3$). This result may be of practical interest. But seeing that the results are obtained in rough assumptions about the concentration of all the gas's mass in an infinitely thin layer behind the shock wave we must define more exactly the area rule.

2. <u>A more exact definition of the area rule.</u> As before we assume that $M_\infty > 1$, but in contrast to another work [1], we do not impose the compulsory condition $M_\infty \tau > 1$. We introduce a cylindrical system of coordinates xL, yL, θ (the x axis passes through the leading point of the body and is directed along the stream). Let us designate by uU_∞, vU_∞, wU_∞ the components of the velocity in the axial, radial, and circular directions respectively; $p\rho_\infty U^2_\infty$ is the pressure, $\rho\rho_\infty$ the density, U_∞ the velocity of the free stream along the <u>x</u> axis, ρ_∞ its density. We will characterize the bluntness value with the dimensionless bluntness diameter dL, where <u>d</u> is a small quantity. We will write the equation of the body surface in the form $y = \tau f(x, \theta)$.

Let us separate the entropic layer, i.e., the region occupied by the lines of flow which have passed that area of the surface of the shock wave where the angles of inclination formed by the surface of the shock wave with the direction of the free flow are not small (Fig. 3). Let the equation of the provisorily interpolated boundary of the entropic layer be $y = \delta\varphi(x, 0)$ where δ is a small quantity. Beginning at a certain $x = x_0 \sim d$ the angles of inclination of the boundary of the entropic layer will have order δ.

Below are listed the estimates of the parameters of the stream in the entropic layer similar to those outlined in another work [3]. Let us also note that the influence of the entropic layer on the pressure distribution on a thin blunt cone is also examined in a work [4].

Let us assume that on the surface of the entropic layer the relationship $p \sim d^{\alpha}$ holds true, where α is a positive number which can be determined.

As results from the following, the order of pressure across the entropic layer is sustained, therefore we may write $\rho \sim d^{\frac{\alpha}{\varkappa}}$ for the density, using the conditions of adiabaticity. Let us now write the continuity equation for the entropic layer. Equating the flow rate in the entropic layer with the flow rate in the stream channel in the free stream across an area equal to the midsection of the bluntness we have

$$d^2 \sim \rho u \sigma, \qquad (1)$$

where σ is the area occupied by the entropic layer in the cross-section where $x = \text{const}$ (hatched in Fig. 3). Insofar as this follows from the Bernouilli equation in the entropic layer $u \sim 1$ we have $\sigma = d^2 = \frac{a}{\varkappa}$. It is obvious that for δ entering the boundary equation of the entropic layer we get the evaluation

$$\delta^2 \sim S + \sigma, \qquad (2)$$

where S is the area of a cross-section of the body. Let us require that the area of the body in its order of magnitude not exceed the entropic layer $S \lesssim \sigma$. Then, obviously

$$\delta^2 \sim d^{2-\frac{\alpha}{\varkappa}}. \tag{3}$$

Since the usual estimate $p \sim \delta^2$ for the hypersonic stream on the exterior boundary of the entropic layer is correct for pressure we derive the equation for determining α:

$$d^\alpha \sim d^{2-\frac{\alpha}{\varkappa}}, \quad \alpha = \frac{2\varkappa}{\varkappa+1} \tag{4}$$

Finally for the stream parameters in the entropic layer we have:

$$p \sim d^{\frac{2\varkappa}{\varkappa+1}}, \quad \rho \sim d^{\frac{2}{\varkappa+1}}, \quad \delta \sim d^{\frac{\varkappa}{\varkappa+1}}, \quad \sigma \sim d^{\frac{2\varkappa}{\varkappa+1}} \tag{5}$$

on condition that τ and d are connected in order of values by the relationship derived from the condition $S \sim \sigma$:

$$\tau \sim d^{\frac{\varkappa}{\varkappa+1}}$$

This relationship practically coincides with the condition $\tau \sim \sqrt{d}$, expressing the fact that body drag is comparable in order of magnitude with bluntness drag [2].

Let us estimate the pressure drop in the entropic layer.

We have from the equation of motion

$$\begin{aligned} \frac{\partial p}{\partial y} &= -\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{w}{y}\frac{\partial v}{\partial \theta}\right), \\ \frac{1}{y}\frac{\partial p}{\partial \theta} &= -\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{w}{y}\frac{\partial w}{\partial \theta}\right). \end{aligned} \tag{7}$$

Consequently $\frac{\partial p}{\partial y} \sim \frac{1}{y}\frac{\partial p}{\partial \theta}$, since all terms on the right side of Eq. 7 are of the same order of smallness. Hence, taking Eqs. 5 and 6 into consideration, the following estimate is correct for the drop in pressure in the radial as well as in the circular direction:

$$\Delta p \sim d^2 \sim \tau^{\frac{2(\varkappa+1)}{\varkappa}}.$$

Thus the pressure in the entropic layer may be considered constant with a relative error

$$\frac{\Delta p}{p} \sim d^{\frac{2}{\varkappa+1}} \sim \tau^{\frac{2}{\varkappa}} \tag{8}$$

somewhat greater than the relative error in the theory of slight disturbances in the hypersonic stream, equalling τ^2 as is well known.

Let us formulate the flow problem. When $x < x_0$ let there be an axisymmetrical nose-piece of a body with an axis of symmetry along the $\underline{x}$ axis, the flow about which is completely calculated. As follows from what has been set forth, since the entropic layer cannot restrain the drop in pressure in the circular direction the pressure in the cross-section where $x = \text{const}$ in the area where $x > x_0$ on the exterior boundary of the layer must be constant. To fulfill this condition it is enough that the surface delimiting the entropic layer have axial symmetry. (Its equation may moreover be written $y = \delta Y(x)$.)

Then the flow outside of the entropic layer, axisymmetrical according to agreement when $x < x_0$, preserves axial symmetry also when $x > x$, and, consequently the stipulation that the pressure be constant in the circular direction on the exterior boundary of the entropic layer will be fulfilled.

Let us derive the equation connecting S and σ when $x > x_0$, for which, as in Sychev [3], we use the continuity equation. We will designate values in the surface x_0 with the subscript 0. Separating the elementary stream channel in the entropy layer we will write this flow-rate equation for it:

$$\rho_0 u_0 y_0 d\theta_0 dy_0 = \rho u y d\theta dy. \tag{9}$$

From adiabaticity equations of Bernouilli, too, who rejects terms of order τ^2, we have for ρ and u:

$$\rho = \rho_0 \left(\frac{p}{p_0}\right)^{\frac{1}{\varkappa}}; \quad \frac{u^2}{2} + \frac{\varkappa}{\varkappa - 1} \frac{p_0}{\rho_0} \left(\frac{p}{p_0}\right)^{\frac{\varkappa - 1}{\varkappa}} = \frac{1}{2} + \frac{1}{(\varkappa - 1) M_\infty^2} \tag{10}$$

Let us analyze Eq. 9 for ρu and integrate over the whole area of the entropy layer in the cross-section $x = x_0$ (we utilize the circumstance that the coordinates of the chosen current lines $\underline{y}$ and θ satisfy the relationships $y = y(y_0, \theta_0)$, $\theta = \theta(y_0, \theta_0)$). On the right side of the equality we obviously will obtain area σ occupied by the entropy layer in cross-section $\underline{x}$. Taking into consideration the axisymmetry of the boundary of the entropy layer we have $\sigma = \pi\theta^2 Y^2(x) - S$. Finally, the desired relationship is written

$$\pi\delta^2 Y^2(x) - F(p) - S(x),$$

$$F(p) = \iint_{\sigma_0} \left(\frac{p_0}{p}\right)^{\frac{1}{\varkappa}} \sqrt{\frac{1 + \frac{2}{\varkappa - 1}\frac{1}{M_\infty^2} - \frac{2\varkappa}{\varkappa - 1}\frac{p_0}{\rho_0}}{1 + \frac{2}{\varkappa - 1}\frac{1}{M_\infty^2} - \frac{2\varkappa}{\varkappa - 1}\frac{p_0}{\rho_0}\left(\frac{p}{p_0}\right)^{\frac{\varkappa - 1}{\varkappa}}}}\, y_0 d\theta_0 dy_0. \qquad (11)$$

Axisymmetrical flow outside the entropy layer when $x > x_0$ may be computed by one of the known exact methods, e.g., by the method of characteristics. The boundary of the entropy layer will be defined moreover in the process of solving from Eq. 11 which plays the role of an edge condition replacing the condition of non-flow. Hence it follows that flow when $x > x_0$ is completely determined by the assignment of a law of change in the area of the cross-section of the body $S(x)$. In this, one must impose one more obvious limitation on the shape of the body: it cannot exceed the limits of the "entropy circle" (see Fig. 3), which may be symbolically written

$$S \subset \pi\delta^2 Y^2(x). \qquad (12)$$

We may now formulate the more precise hypersonic area rule in the following way. When thin blunt bodies having axisymmetrical nosepieces coinciding for some distance from the leading point of the body and having also the same laws of change in the areas of the cross-section of the remaining parts:

a) flows outside of the entropy layers are axisymmetrical; the parameters of the flows at corresponding points, the surfaces of the shock waves, and the provisorily introduced boundaries of the entropy layers coincide;

b) pressure in the entropy layers depends only on $\underline{x}$ and the law of pressure change is the same for the bodies under investigation; as a result of this the resistance forces acting on the bodies are equal as the resistance X is expressed in the form

$$X = X_0 + L^2 \rho_\infty U_\infty^2 \int_{x_0}^{1} S'(x)\, p(x)\, dx, \tag{13}$$

where X_0 is the resistance of the nosepiece of the body. In addition it is assumed that conditions 6 and 12 are fulfilled.

The results obtained are easily generalized to the case of flows with dissociation. A consideration of these phenomena leads only to a change in the form of function F(p) in Eq. 11.

3. Comparison of the results. Let us compare the result we have obtained with the area rule proven in another work [1]. The requirement for the coincidence of the laws of change in area of a cross-section in the direction of the $\underline{x}$ axis and also the stipulation that body drag must not exceed bluntness drag in order of magnitude are common to both theorems. The difference in the theorem proven in section 2 in the formulation consists in the following:

a) the obligatory condition $M_\infty \tau > 1$ is not imposed as is done in the other work [1];

b) instead of the requirement for equality in the magnitudes of bluntness resistance [1] a more severe limitation is imposed: the nosepieces of the equivalent bodies must coincide in form, being axisymmetrical;

c) instead of condition 3 in the work [1], according to which the body does not exceed the volume delimited by the surface of the shock wave, the severer limitation 12 is imposed that the body must not exceed the limits of the inner boundary of the entropy layer.

As a result of this we may expect that the values of k found in section 1 characterizing the difference of cross-section of the body from the cross-section of the equivalent body of revolution are somewhat too high.

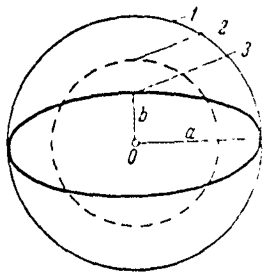


Fig. 1.

1) Shock wave; 2) cross section of the circular body; 3) cross section of the equivalent body.

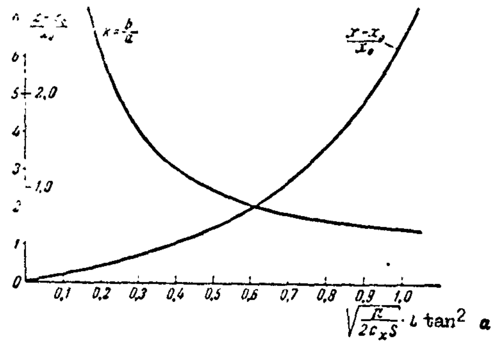
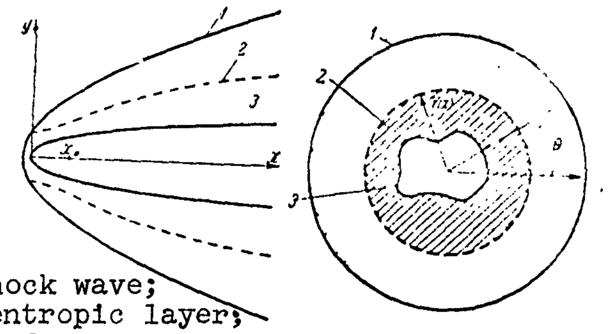


Fig. 2.



1) Shock wave;
2) entropic layer;
3) body

Fig. 3.

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